Matrix Inverse

$A \cdot A^{-1} = I$

calculating the inverse.

use diff. b's.

if b is $\begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}$, the resulting solution is the first column of matrix inverse.

similarly $\begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix}$, $\begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}$

Procedure

$Ax = b$

to find A^{-1}

solve $Ax = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}$ ✓

$Ax = \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix}$ ✓

$Ax = \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}$ ✓

effort required is $\frac{n^3}{3}$.

Stimulus Response.

interaction response stimuli

$Ax = b$

$x = A^{-1} b$

$x_1 = \underbrace{a^{-1}_{11}} \underbrace{b_1} + a^{-1}_{12} b_2 + a^{-1}_{13} b_3$

Error Analysis and System condition

Inverse provides a means to tell if the system is ill conditioned.

1. Scale the matrix A, so that Largest element in each row is 1.
   Invert the scaled matrix.
   if there are elements of A^{-1} >>> 1
   the system is likely to be ill conditioned.

2. Multiply the inverse with original matrix
   $A^{-1} \cdot A$ = check if it is I or nA.

3. Find inverse of inverse
   $[A^{-1}]^{-1} = A$ check if it is true.

Condition number

Norm

A norm is a real valued function that provides a measure of the size or length of multi component mathematical entities.

$\vec{F} = [ai + bj + ck]$

length of this vector $= \sqrt{a^2+b^2+c^2}$

Euclidean norm of F $= \|F\|_e = \sqrt{a^2+b^2+c^2}$

For an n-dimensional vector

$$X = [x_1, x_2, x_3, \ldots\ldots x_n]$$

$$\|X\|_e = \sqrt{\sum_{i=1}^{n} x_i^2}$$

for a matrix A

$$\|A\|_e = \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} a_{i,j}^2}$$

{ frobenius norm }

<u>uniform vector norm</u>

$$\|X\|_\infty = \max_{1 \le i \le n} |x_i|$$

element with Largest abs. value

<u>uniform matrix norm</u> (row-sum norm)

$$\|A\|_\infty = \max_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}|$$

{ sum of abs. value in a row & the Largest is taken

<u>Matrix Condition Number</u>

$$\text{Cond}[A] = \|A\| \cdot \|A^{-1}\|$$

This number will be greater than 1. ≥ 1

$$\frac{\|\Delta X\|}{\|X\|} \leq \text{Cond}[A] \frac{\|\Delta A\|}{\|A\|}$$

if Cond $[A] >>> 1$, then matrix is ill conditioned

relative error of the norm of computed solution be as large as relative error of norm of

$$\frac{\|\Delta X\|}{\|X\|} \leq \text{Cond}[A] \frac{\|\Delta A\|}{\|A\|}$$

relative error of the norm of the computed solution can be as large as the relative error of the norm of the coefficients [A] multiplied by the condition number.

ex. if [A] are know to $\frac{\text{t-digit Precision}}{\text{rounding off err} = \bar{1}}$

and cond $[A] = 10^{c}$

s^n [X] may be valid to <u>only t-c digits</u>

rounding error 10^{c-t}

<u>Ex</u>. Hilbert matrix, 3^{rd} order,

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots & \frac{1}{n+1} \\ \vdots & & & & \\ \frac{1}{n} & \frac{1}{n+1} & \frac{1}{n+2} & \cdots & \frac{1}{2n-1} \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{bmatrix}$$

Sol^n scaling

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ 1 & \frac{2}{3} & \frac{1}{2} \\ 1 & \frac{3}{4} & \frac{3}{5} \end{bmatrix} \quad \begin{matrix} 1.833 \\ 2.167 \\ 2.35 \end{matrix}$$

find norm $\|A\|_\infty = 2.35$

$$A^{-1} = \begin{bmatrix} 9 & -18 & 10 \\ -36 & 96 & -60 \\ 30 & -90 & 60 \end{bmatrix} \quad \begin{matrix} 37 \\ 192 ✓ \\ 180 \end{matrix}$$

$\|A^{-1}\|_\infty = 192$

$\text{Cond}[A] = 2.35 \times 192 = 451.2$

system is ill conditioned.

extent of ill conditioning

$C = \log 451.2 = 2.65$

computer t = , $\log 2^{-24} = 7.2$

rounding off error $= 10^{2.65 - 7.2} = 3 \times 10^{-5}$

Iterative refinement

$$A x = b$$

$$A x_a = b_a$$

$$A \Delta x = b - b_a$$

$$\boxed{x_a + \Delta x}$$ repeat.